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A SYSTEM OF EQUATIONS FOR OPTIMIZED POWERED FLIGHT TRAJECTORIES

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ABSTRACT

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DEFINITION OF SYMBOLS

Symbol	Definition and Units
F_v	vacuum thrust (kg)
A_e	engine exit area (m^2)
A	reference area (m^2)
I_{sp_V}	vacuum specific impulse (sec)
C_{Do}	drag coefficient
C'_N	normal force coefficient
ψ	launch latitude (deg)
A_z	launch azimuth (deg)
ω	earth's rotational velocity in the equatorial plane (rad/sec)
ω'	earth's rotational velocity in the flight plane (rad/sec)
P	atmospheric pressure
ρ	atmospheric density
F	thrust (kg)
D	drag force (kg)
N	normal force (kg)
g	gravitational acceleration (m/sec^2)
GM	gravitational constant (m^3/sec^2)
r	radius from center of earth to vehicle (m)
V_E	earth-fixed velocity (m/sec)

DEFINITION OF SYMBOLS (Continued)

Symbol	Definition and Units
ϑ_E	earth-fixed path angle measured from local vertical to velocity vector (deg)
φ_E	range angle measured from earth-fixed launch point to radius vector (deg)
m	mass of vehicle (kg-sec^2/m))
α	angle of attack (deg)
$\dot{\alpha}$	angle of attack rate (deg/sec)
R_o	earth's radius (m)
g_o	gravitational acceleration at the surface of the earth (m/sec^2)
M	Mach
χ	attitude angle measured from the earth-fixed launch point to the vehicle's thrust vector (deg)
χ^*	space-fixed attitude angle (deg)
φ^*	space-fixed range angle (deg)
XXX_E	earth-fixed range (m)
XXX^*	space-fixed range (m)
λ, μ	Lagrangian multipliers
t	time (sec)
$\bar{X}^*$	space-fixed coordinate system
$\bar{X}_E$	earth-fixed coordinate system
$(\dot{\ })$	d/dt
$(\ )_o$	initial quantity

DEFINITION OF SYMBOLS (Cont'd)

Symbol	Definition and Units
$(\ )_f$	final quantity
$(\bar{\ })$	vector quantity
$\bar{i}$, $\bar{j}$, $\bar{k}$, $\bar{\ell}$, $\bar{e}_V$, $\bar{e}_\vartheta$	unit vectors defined in Section II
C_1	constant to convert mass units to pounds
C_2	constant to convert kilograms to pounds

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A SYSTEM OF EQUATIONS FOR OPTIMIZED POWERED FLIGHT TRAJECTORIES

SUMMARY

The equations of motion for a vehicle with thrust, lift, and drag forces, and a Newtonian gravitational force are derived in an earth-fixed polar coordinate system. This system of equations forms the differential equations of constraint in the calculus of variations formulation of minimizing flight time between two sets of boundary conditions with inequality constraints imposed on the magnitude of the angle of attack or the product of the dynamic pressure and the magnitude of the angle of attack. The necessary conditions for optimality are given exclusive of derivation. Also, a computational scheme is given suitable for a digital computer program.

I. INTRODUCTION

Normally while in the atmosphere a vehicle is constrained to fly a non-lifting trajectory, after tilting has been initiated shortly after launch, in order to minimize the structural stresses associated with appreciable angles of attack. In essence, this implies that the aerodynamic lift is sacrificed; this may decrease the performance of the vehicle depending on the lift-to-drag ratio. The problem of employing an angle of attack during an atmospheric flight can be used if constraints are placed on functions related to the structural stresses on the vehicle. This type of problem is presented in this paper by the calculus of variations technique where the constraint functions related to these stresses are explicit functions of the control variable, the angle of attack, α.

The purpose of this paper then is to derive the equations of motion in an atmospheric flight and to formulate an optimization technique with inequality constraints imposed on two functions related to structural stresses encountered during the atmospheric flight. The two functions chosen in this paper are the product of the dynamic pressure and the magnitude of the angle of attack and the magnitude of the angle of attack alone.

II. DERIVATION OF THE EQUATIONS OF MOTION

When calculating a space vehicle's trajectory in the atmosphere, it is appropriate to derive the equations of motion in a coordinate system fixed in (rotating with) the earth because the earth's atmosphere is assumed to rotate uniformly with the earth and the external forces acting on the vehicle (thrust, lift and drag) are measured relative to the rotating atmosphere. Also, the centrifugal and Coriolis acceleration of a rotating earth affect the vehicle's motion.

In this paper the equations of motion are derived in a two-dimensional polar coordinate system. To simulate a rotating earth, it is assumed that this coordinate system rotates with an angular velocity relative to some inertial coordinate system. This angular velocity is measured at the launch point and is a function of the latitude and aiming azimuth at launch.

In deriving the equations of motion, it is convenient to define two Cartesian coordinate systems. The first of these systems (denoted by $\bar{X}^*$) is space-fixed with its Y^*-axis directed from the center of the earth through the launch point and with its X^*-axis directed along and parallel to the launch azimuth. The second system (denoted by $\bar{X}_E$) is earth-fixed with its coordinate axes coinciding with the space-fixed system at the time of launch. This system rotates with an angular velocity given by

$$\omega' = \omega \cos \psi \sin A_z \tag{2.1}$$

relative to the space-fixed system. In the above equation, ω is the earth's rotational velocity in the equatorial plane, ψ is the latitude above the equatorial plane, and A_z is the aiming azimuth.

First, the space-fixed velocity vector is derived in terms of the earth-fixed Cartesian coordinate system. A transformation is then made from the earth-fixed Cartesian system to a polar coordinate system which is also fixed in the earth. The space-fixed acceleration vector is then derived in terms of the polar coordinate system. Finally, by Newton's second law the external forces acting on the vehicle are equated to the mass times the acceleration. Taking the two components of this vector equation yields the two scalar equations for acceleration.

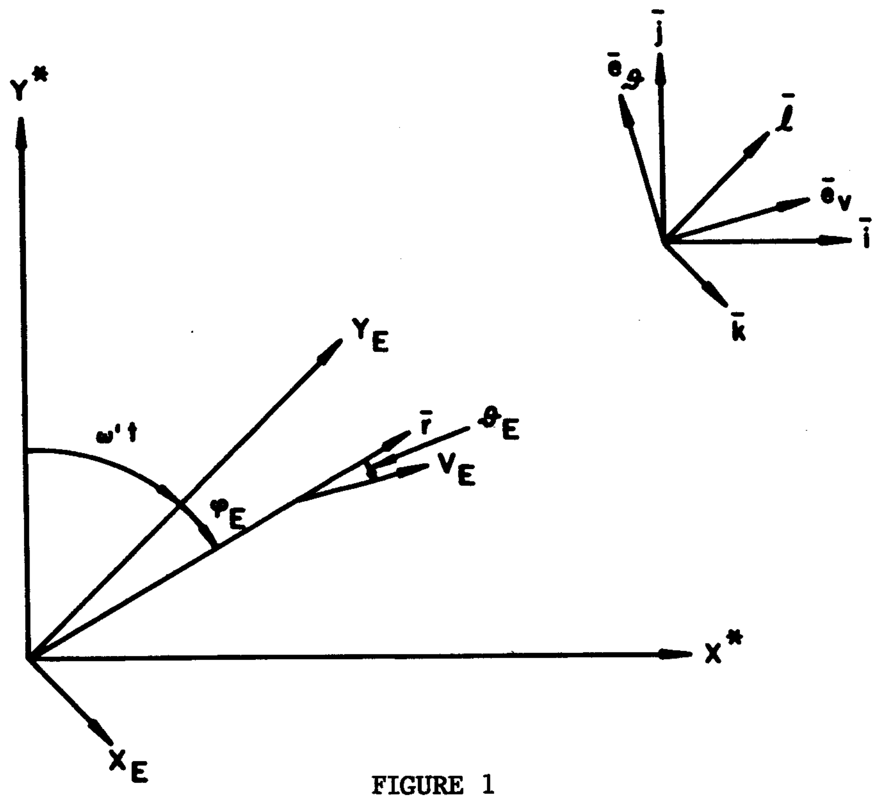

FIGURE 1

In Figure 1, $\bar{i}$ and $\bar{j}$ are unit vectors in the space-fixed system with $\bar{i}$ and $\bar{j}$ being parallel to the X^* and Y^* axis, respectively. The unit vectors $\bar{k}$ and $\bar{\ell}$ are parallel to the X_E and Y_E axis, respectively, in the earth-fixed system. The unit vector $\bar{e}_v$ is defined to be parallel to the earth-fixed velocity vector and $\bar{e}_\vartheta$ is defined to be in the opposite direction of the increasing earth-fixed flight path angle, ϑ_E. The relationship between $\bar{k}$ and $\bar{\ell}$ and the unit vectors in the space-fixed system is given by

$$\bar{k} = \cos \omega' t \; \bar{i} - \sin \omega' t \; \bar{j} \tag{2.2}$$

$$\bar{\ell} = \sin \omega' t \; \bar{i} + \cos \omega' t \; \bar{j}. \tag{2.3}$$

Similarly, the relationship between $\bar{k}$ and $\bar{\ell}$ and the unit vectors in the polar coordinate system is given by

$$\bar{k} = \sin (\varphi_E + \vartheta_E) \; \bar{e}_v - \cos (\varphi_E + \vartheta_E) \; \bar{e}_\vartheta \tag{2.4}$$

$$\bar{\ell} = \cos (\varphi_E + \vartheta_E) \; \bar{e}_v + \sin (\varphi_E + \vartheta_E) \; \bar{e}_\vartheta. \tag{2.5}$$

The position vector in terms of the earth-fixed Cartesian system is

$$\bar{r} = X_E \bar{k} + Y_E \bar{\ell}. \tag{2.6}$$

The time derivative of (2.6) in the space-fixed system gives the space-fixed velocity vector in terms of the earth-fixed position and velocity components

$$\dot{\bar{r}} = \dot{X}_E \bar{k} + \dot{Y}_E \bar{\ell} + X_E \dot{\bar{k}} + Y_E \dot{\bar{\ell}}, \tag{2.7}$$

where the time derivatives of the unit vectors can be obtained from (2.2) and (2.3) which are

$$\dot{\bar{k}} = - \omega' \sin \omega' t \; \bar{i} - \omega' \cos \omega' t \; \bar{j} = - \omega' \; \bar{\ell} \tag{2.8}$$

$$\dot{\bar{\ell}} = \omega' \cos \omega' t \; \bar{i} - \omega' \sin \omega' t \; \bar{j} = \omega' \; \bar{k}. \tag{2.9}$$

Substituting these values into (2.7) gives the space-fixed velocity vector

$$\dot{\bar{r}} = (\dot{X}_E + \omega' Y_E) \; \bar{k} + (\dot{Y}_E - \omega' X_E) \; \bar{\ell}. \tag{2.10}$$

The position and velocity components in (2.10) in the earth-fixed polar coordinate system are

$$X_E = r \sin \varphi_E \tag{2.11}$$

$$Y_E = r \cos \varphi_E \tag{2.12}$$

$$\dot{X}_E = V_E \sin (\varphi_E + \vartheta_E) \tag{2.13}$$

$$\dot{Y}_E = V_E \cos (\varphi_E + \vartheta_E). \tag{2.14}$$

Substituting back into (2.10) gives the space-fixed velocity vector, in terms of the earth-fixed polar coordinate system components, which becomes

$$\dot{\bar{r}} = \left\{V_E \sin(\varphi_E + \vartheta_E) + \omega' r \cos\varphi_E\right\} \bar{k} + \left\{V_E \cos(\varphi_E + \vartheta_E) - \omega' r \sin\varphi_E\right\} \bar{\ell} \tag{2.15}$$

Taking the time derivative of (2.15) in the space-fixed system and using (2.4) and (2.5) gives the space-fixed acceleration vector

$$\begin{aligned}
\ddot{\bar{r}} = \Big[\dot{V}_E & \left\{\sin^2(\varphi_E + \vartheta_E) + \cos^2(\varphi_E + \vartheta_E)\right\} + \omega' \dot{r} \left\{\sin(\varphi_E + \vartheta_E) \cos\varphi_E \right.\\
& \left. - \cos(\varphi_E + \vartheta_E) \sin\varphi_E\right\} - \omega' r \dot{\varphi}_E \left\{\sin(\varphi_E + \vartheta_E) \sin\varphi_E \right.\\
& \left. + \cos(\varphi_E + \vartheta_E) \cos\varphi_E\right\} - \omega'^2 r \left\{\sin(\varphi_E + \vartheta_E) \sin\varphi_E \right.\\
& \left. + \cos(\varphi_E + \vartheta_E) \cos\varphi_E\right\}\Big] \bar{e}_v + \Big[-V_E(\dot{\varphi}_E + \dot{\vartheta}_E) \left\{\sin^2(\varphi_E + \vartheta_E) \right.\\
& \left. + \cos^2(\varphi_E + \vartheta_E)\right\} - \omega' \dot{r} \left\{\sin(\varphi_E + \vartheta_E) \sin\varphi_E \right.\\
& \left. + \cos(\varphi_E + \vartheta_E) \cos\varphi_E\right\} - \omega' r \dot{\varphi}_E \left\{\sin(\varphi_E + \vartheta_E) \cos\varphi_E \right.\\
& \left. + \cos(\varphi_E + \vartheta_E) \sin\varphi_E\right\} - \omega' V_E \left\{\sin^2(\varphi_E + \vartheta_E) + \cos^2(\varphi_E + \vartheta_E)\right\}\\
& - \omega'^2 r \left\{\sin(\varphi_E + \vartheta_E) \cos\varphi_E + \cos(\varphi_E + \vartheta_E) \sin\varphi_E\right\}\Big] \bar{e}_\vartheta .
\end{aligned} \tag{2.16}$$

In the earth-fixed polar coordinate system $\dot{r}$ and $\dot{\varphi}_E$ are

$$\dot{r} = V_E \cos \vartheta_E \tag{2.17}$$

$$\dot{\varphi}_E = \frac{V_E}{r} \sin \vartheta_E . \tag{2.18}$$

Substituting these equations into (2.16) and after simplification (2.16) becomes

$$\ddot{\bar{r}} = \left[\dot{V}_E - \omega'^2 r \cos \vartheta_E\right] \bar{e}_v + \left[V_E \dot{\vartheta}_E + \left(\frac{V_E^2}{r} + \omega'^2 r\right) \sin \vartheta_E + 2\omega' V_E\right](-\bar{e}_\vartheta). \tag{2.19}$$

This equation gives the two components of acceleration measured in the earth-fixed polar coordinate system. The second component is in the direction of the increasing flight path angle since $+\bar{e}_\vartheta$ was defined in the opposite direction.

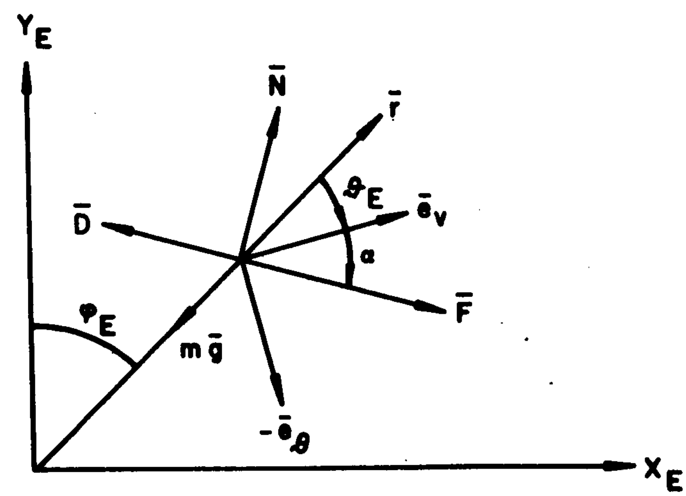

FIGURE 2

Assuming the vehicle as a point mass subjected to the external forces in Figure 2, the accelerations caused by these forces are equated to the like components in equation (2.19) by Newton's second law. This yields the following two scalar differential equations

$$\dot{V}_E = \frac{F - D}{m} \cos \alpha - \frac{N}{m} \sin \alpha - (g - \omega'^2 r) \cos \vartheta_E \tag{2.20}$$

$$\dot{\vartheta}_E = \frac{F - D}{mV_E} \sin \alpha + \frac{N}{mV_E} \cos \alpha + \frac{1}{V_E}\left(g - \frac{V_E^2}{r} - \omega'^2 r\right) \sin \vartheta_E - 2\omega' , \tag{2.21}$$

where

$$F = F_v - A_e P \tag{2.22}$$

$$D = \frac{1}{2} \rho V_E^2 C_{Do} A = q C_{Do} A \tag{2.23}$$

$$N = \frac{1}{2} \rho V_E^2 C_N' A\alpha = q C_N' A\alpha = N_4 \alpha \tag{2.24}$$

$$g = \frac{GM}{r^2} , \tag{2.25}$$

where P and ρ are determined from some atmospheric model, ARDC or Patrick for example. By instantaneously considering these quantities as exponential functions given by

$$P = P_o e^{-\gamma (r - R_o)} \tag{2.26}$$

$$\rho = \rho_o e^{-\mu (r - R_o)} , \tag{2.27}$$

γ and μ can be obtained by taking the inverse of (2.26) and (2.27), respectively. The partial derivatives $\partial P/\partial r$ and $\partial \rho/\partial r$ which are needed in the optimization equations would be

$$\frac{\partial P}{\partial r} = - \gamma P \tag{2.28}$$

$$\frac{\partial \rho}{\partial r} = - \mu \rho . \tag{2.29}$$

C_{Do} and C_N' in equations (2.23) and (2.24) are considered as functions of Mach number, and their derivatives are ignored in the optimization problem.

The equations (2.17), (2.18), (2.19), (2.20) and the equation for fuel flow rate,

$$\dot{m} = - \frac{F_V}{g_o \; I_{sp_V}} = \text{constant}, \tag{2.30}$$

will form the system of differential equations of motion that have to be solved simultaneously to determine a trajectory. The next section will discuss the method of determining the optimum control function, the angle of attack as a function of time, such that a trajectory minimizes time between two sets of boundary conditions subject to equality and inequality constraints.

III. CALCULUS OF VARIATIONS FORMULATION

Normally while in the atmosphere a vehicle is constrained to fly a non-lifting trajectory, after tilting has been initiated shortly after launch, in order to minimize the structural stresses associated with appreciable angles of attack. In essence this implies that the aerodynamic lift is sacrificed which may decrease the performance of the vehicle depending on the lift to drag ratio. The problem of employing an angle of attack during an atmospheric flight can be used if constraints are placed on functions related to structural stresses on the vehicle. This type of problem can be treated by the calculus of variations technique where the constraint functions related to these stresses are explicit functions of the control variable, the angle of attack, α.

Two such constraint functions are used in this paper:

$$g_1(x, \alpha) = QX - q|\alpha| \geqq 0 \tag{3.1}$$

$$g_2(\alpha) = \alpha_c^2 - \alpha^2 \geqq 0, \tag{3.2}$$

where QX and α_c are numbers depending on the design limits of the vehicle and x denotes the set of state variables $(r, V_E, \vartheta_E, \varphi_E, m)$. The first equation (3.1) implies a constraint on the product of the dynamic pressure, q, and the magnitude of the angle of attack, whereas (3.2) constrains the magnitude of the angle of attack.

The variational problem may now be stated as follows: It is desired to determine the control function, $\alpha(t)$, such that the expression

$$M = \int_{t_o}^{t_f} f_o(x, \alpha)\, dt + \Phi[x_f, t_f] \tag{3.3}$$

is maximized subject to the inequality constraints (3.1) and (3.2) and the differential equations of constraint

$$\dot{x}_i = f_i(x, \alpha, t) \qquad i = 1, \ldots, n \tag{3.4}$$

with the boundary conditions

$$x_i(t_o) = x_{io} \tag{3.5}$$

and

$$\psi_\ell(x_f, t_f) = E_\ell \qquad \ell = 1, \ldots, q, \tag{3.6}$$

where the ψ_ℓ's are the terminal constraint functions on the state variables.

In order that (3.3) is maximized the following necessary conditions must hold (see Reference 1):

$$\mu_k \leq 0 \qquad k = 1, 2 \tag{3.7}$$

$$\dot{\lambda}_i = -\frac{\partial F'}{\partial x_i} \qquad i = 1, \ldots, n \tag{3.8}$$

$$\frac{\partial F'}{\partial \alpha} = 0, \tag{3.9}$$

where

$$F' = f_o(x, \alpha) + \sum_{i=1}^{n} \lambda_i(t)\left\{f_i(x,\alpha,t) - \dot{x}_i\right\} + \sum_{k=1}^{2} \mu_k\, g_k(x, \alpha), \tag{3.10}$$

and the boundary conditions

$$\lambda_i(t_f) = \frac{\partial \Phi}{\partial x_i} - \sum_{\ell=1}^{q} \nu_\ell \left(\frac{\partial \psi_\ell}{\partial x_i}\right)_{t=t_f} \tag{3.11}$$

$$f_o(x_f, \alpha_f) + \sum_{i=1}^{n} \lambda_i(t_f)\, f_i(x_f, \alpha_f) = \sum_{\ell=1}^{q} \nu_\ell \frac{\partial \psi_\ell}{\partial t_f} - \frac{\partial \Phi}{\partial t_f}, \tag{3.12}$$

where the λ_i's and μ_k's are Lagrangian multiplier variables and the ν_ℓ's are multiplier constants evaluated at t_f.

Equation (3.8) can be written as

$$\frac{\partial F'}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial F'}{\partial \dot{x}_i}\right) = 0; \tag{3.13}$$

then

$$\sum_{i=1}^{N} \left[\frac{\partial F'}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial F'}{\partial \dot{x}_i}\right)\right] = \frac{d}{dt}\left[- F' + \sum_{i=1}^{n} \frac{\partial F'}{\partial \dot{x}_i}\, \dot{x}_i\right] + \frac{\partial F'}{\partial t} = 0. \tag{3.14}$$

If F' is not an explicit function of time, then (3.14) becomes

$$- F' + \sum_{i=1}^{n} \frac{\partial F'}{\partial \dot{x}_i}\, \dot{x}_i = - C = \text{constant}. \tag{3.15}$$

Then, by equations (3.8) and (3.10), equation (3.15) becomes

$$f_o(x, \alpha) + \sum_{i=1}^{n} \lambda_i(t)\, f_i(x, \alpha) = C. \tag{3.16}$$

If the end conditions (3.6) are not explicit functions of time, then

$$C = - \frac{\partial \Phi}{\partial t_f}, \tag{3.17}$$

since (3.16) is to be satisfied from t_o to t_f.

If we wish to minimize flight time between the boundary conditions (3.5) and (3.6) for a vehicle whose motion is governed by the differential equations of motion in part I, the expression (3.3) reduces to maximizing

$$\Phi[x_f, t_f] = - t_f \tag{3.18}$$

and the function $f_o(x, \alpha)$ is zero. Maximizing $-t_f$ is equivalent to minimizing t_f. Equation (3.10) becomes

$$F' = \sum_{i=1}^{n} \lambda_i(t)\left\{f_i(x,\alpha)-\dot{x}_i\right\}+\sum_{k=1}^{2} \mu_k\, g_k(x, \alpha), \tag{3.19}$$

where the constraining equations (3.4) are given by

$$\dot{\phi}_E = \frac{V_E}{r} \sin \vartheta_E = f_1 \tag{3.20}$$

$$\dot{r} = V_E \cos \vartheta_E = f_2 \tag{3.21}$$

$$\dot{V}_E = \frac{F - D}{m} \cos \alpha - \frac{N}{m} \sin \alpha - (g - \omega'^2 r) \cos \vartheta_E = f_3 \tag{3.22}$$

$$\dot{\vartheta}_E = \frac{F - D}{mV_E} \sin \alpha + \frac{N}{mV_E} \cos \alpha + \frac{1}{V_E}\left(g - \frac{V_E^2}{r} - \omega'^2 r\right) \sin \vartheta_E - 2\omega' = f_4 \tag{3.23}$$

$$\dot{m} = - \frac{F_v}{g_o\, I_{sp_v}} = f_5, \tag{3.24}$$

and the inequality constraints are given by (3.1) and (3.2). The terminal constraints (3.6) of the problem are given by

$$\psi_1 = r_f - r(t_f) = 0 \tag{3.25}$$

$$\psi_2 = V_f - V_E(t_f) = 0 \tag{3.26}$$

$$\psi_3 = \vartheta_f - \vartheta_E(t_f) = 0, \tag{3.27}$$

where r_f, V_f, and ϑ_f are the desired end conditions.

Applying equation (3.8) to (3.19) and assuming that (3.1) and (3.2) are satisfied results in the following system of differential equations:

$$\dot{\lambda}_1 = - \frac{\partial F'}{\partial \varphi_E} = 0 \tag{3.28}$$

$$\begin{aligned}\dot{\lambda}_2 = - \frac{\partial F'}{\partial r} = &- \frac{\lambda_3}{m}\left[\mu N \sin \alpha + \frac{2m}{r} g \cos \vartheta_E + \left\{\gamma(F_v - F) + \mu D\right\} \cos \alpha \right. \\ &\left. + m\omega'^2 \cos \vartheta_E\right] - \frac{\lambda_4}{mV_E}\left[\left\{\gamma(F_v - F) + \mu D\right\} \sin \alpha - \mu N \cos\alpha \right. \\ &\left. + \frac{m}{r}\left(\frac{V_E^2}{r} - 2g - \omega'^2 r\right) \sin \vartheta_E\right]\end{aligned} \tag{3.29}$$

$$\dot{\lambda}_3 = -\frac{\partial F'}{\partial V_E} = -\lambda_2 \cos \vartheta_E + \frac{2\lambda_3}{mV_E}\left[N \sin \alpha + D \cos \alpha\right]$$

$$+ \frac{2\lambda_4}{mV_E^2}\left[D \sin \alpha - N \cos \alpha + \frac{mV_E^2}{r} \sin \vartheta_E\right] \tag{3.30}$$

$$+ \frac{\lambda_4}{V_E}\left[\frac{1}{mV_E}\left\{(F - D) \sin \alpha + N \cos \alpha + m\left(g - \frac{V_E^2}{r} - \omega'^2 r\right) \sin \vartheta_E\right\}\right]$$

$$\dot{\lambda}_4 = -\frac{\partial F'}{\partial \vartheta_E} = \lambda_2 V_E \sin \vartheta_E + \lambda_3 \left[(\omega'^2 r - g) \sin \vartheta_E\right]$$

$$+ \lambda_4 \left[\frac{1}{V_E}\left(\frac{V_E^2}{r} - g + \omega'^2 r\right) \cos \vartheta_E\right] \tag{3.31}$$

$$\dot{\lambda}_5 = -\frac{\partial F'}{\partial m} = \frac{\lambda_3}{m^2}\left[(F - D) \cos \alpha - N \sin \alpha\right] + \frac{\lambda_4}{m^2 V_E}\left[(F - D) \sin \alpha + N \cos \alpha\right]. \tag{3.32}$$

Applying (3.9) to (3.14) yields the equation

$$\frac{\partial F'}{\partial \alpha} = f_\alpha = -\left[\lambda_3 (F - D + N_4) + \lambda_4 \frac{N_4}{V_E} \alpha\right] \sin \alpha$$

$$+ \left[\frac{\lambda_4}{V_E} (F - D + N_4) - \lambda_3 N_4 \alpha\right] \cos \alpha = 0. \tag{3.33}$$

The angle of attack is determined from the iterative solution of (3.33). If at some time the angle of attack does not satisfy the inequality (3.1), then a $q|\alpha|$ constraint exists. During this constraint period, the angle of attack is computed by

$$\alpha^* = \frac{QX \cdot \alpha}{q|\alpha|}, \tag{3.34}$$

where $\frac{\alpha}{|\alpha|}$ determines the sign of α^*. The multiplier μ_1 during the constraint is given by

$$\mu_1 = \left[f_\alpha\right]_{\alpha=\alpha^*} \div m\left[-\frac{\partial g_1}{\partial \alpha}\right]_{\alpha=\alpha^*}. \tag{3.35}$$

Equations (3.29) and (3.30) are augmented by

$$-\mu_1 \frac{\partial g_1}{\partial q} \cdot \frac{\partial q}{\partial r}$$

$$-\mu_1 \frac{\partial g_1}{\partial q} \cdot \frac{\partial q}{\partial V_E},$$

respectively. The constraint ends once μ_1 goes to zero.

If the inequality (3.2) is not satisfied by the angle of attack from (3.33), the angle of attack becomes

$$\alpha = \pm\, \alpha_c, \tag{3.36}$$

where α_c assumes the sign of α determined from (3.33). During the constraint μ_2 is given by

$$\mu_2 = \left[f_\alpha\right]_{\alpha=\alpha_c} \div m\left[-\frac{\partial g_2}{\partial \alpha}\right]_{\alpha=\alpha_c}. \tag{3.37}$$

Again, the constraint ends once μ_2 goes to zero.

It must be noted here that both inequality constraints cannot be considered simultaneously. If, for example, (3.1) is considered, then μ_2 would be zero; or vice versa, if (3.2) is considered.

The problem remains now of selecting the initial values of the Lagrangian multipliers such that all of the necessary conditions and the boundary conditions (3.25 - 3.27) are satisfied. First, it is noted that $\lambda_1(t)$ is a constant by equation (3.28). Furthermore, since $\varphi_E(t_f)$ is allowed to vary in the problem presented in this paper, $\lambda_1(t_f)$ is zero by virtue of (3.11). Since $f_o(x, \alpha)$ is zero in equation (3.10), the system of equations (3.29 - 3.32) is homogeneous in the λ's. This property makes one of the initial values of the λ's arbitrary. In this paper, we have chosen

$$\lambda_3(t_o) = \lambda_{3o} = 1. \tag{3.38}$$

By specifying an initial angle of attack, α_o, and using (3.33), the initial value of λ_4 is given by

$$\lambda_{4o} = -\left(\frac{\partial \dot{V}_E}{\partial \alpha}\right) \div \left(\frac{\partial \dot{\vartheta}_E}{\partial \alpha}\right). \tag{3.39}$$

Furthermore, by specifying an initial angle of attack rate, $\dot{\alpha}_o$, and differentiating (3.33) with respect to time and using (3.28), (3.29), λ_{3o}, and λ_{4o}, λ_{2o} can be determined. Since equations (3.29 - 3.32) are independent of λ_5, the initial value of λ_5 can be taken as zero.

Thus, α_o and $\dot{\alpha}_o$ are used to isolate the terminal constraints (3.25 - 3.27). Once these terminal constraints have been satisfied the terminal values of the λ's are by (3.11) and (3.25 - 3.27)

$$\lambda_2(t_f) = \nu_1, \; \lambda_3(t_f) = \nu_2, \text{ and } \lambda_4(t_f) = \nu_3.$$

Since, in equation (3.12) the ψ_ℓ's are not explicit functions of time, (3.12) is given by

$$\sum_{i=1}^{n} \lambda_i(t_f)\, f_i(x_f, \alpha_f) = -\frac{\partial \Phi}{\partial t_f} = \text{const}, \tag{3.40}$$

which becomes (3.16) evaluated at t_f.

IV. APPLICATIONS FOR A DIGITAL COMPUTER PROGRAM

In this section, a computational scheme is given suitable for a digital computer program. First, the inequality constraints on $q|\alpha|$ and α are ignored. This is the basic program, and only slight modifications are needed to include the inequality constraints which are given after the basic scheme.

A. Application Without Inequality Constraints

1. Input Data Needed

Constants: g_o, R_o, C_1, C_2, A_z, ψ, $\omega, \Delta t$,

Initial Conditions: φ_{E_o}, r_o, V_{E_o}, ϑ_{E_o}

Tables: C_{Do}, C_N', Mach

Isolation Parameters: α_o, $\dot{\alpha}_o$

Vehicle Data: F_v(lbs), I_{sp_v}(sec), $A(m^2)$, $A_e(m^2)$, W_o(lbs).

2. Preload Computations

$$m_o = W_o \div C_1 \tag{4.1}$$

$$\dot{m} = -\frac{F_v}{C_2 \, I_{sp_v}} \tag{4.2}$$

$$F_v(kg) = F_v(lbs) \div C_2 \tag{4.3}$$

$$\omega' = \omega \sin A_z \cos \psi \tag{4.4}$$

$$GM = g_o R_o^2. \tag{4.5}$$

To compute λ_{20} and λ_{40}, let

$$J_4 = - \frac{\partial \dot{V}_E}{\partial \alpha} \tag{4.6}$$

$$K_4 = - \frac{\partial \dot{\vartheta}_E}{\partial \alpha} . \tag{4.7}$$

Then equation (3.33) becomes

$$f_\alpha = \lambda_3 \, J_4 + \lambda_4 \, K_4 = 0 \tag{4.8}$$

at t_o: $\lambda_{30} = 1$, $\alpha = \alpha_o$.

Then,

$$\lambda_{40} = - K_4 \div J_4. \tag{4.9}$$

Taking the time derivative of (4.8) gives

$$\dot{\lambda}_3 \, J_4 + \lambda_3 \, \dot{J}_4 + \dot{\lambda}_4 \, K_4 + \lambda_4 \, \dot{K}_4 = 0. \tag{4.10}$$

To determine $\dot{J}_4$ and $\dot{K}_4$, we compute:

$$\mu = \frac{1}{r - R_o} \, \ln \frac{\rho_o}{\rho} \tag{4.11}$$

$$\gamma = \frac{1}{r - R_o} \, \ln \frac{P_o}{P} \tag{4.12}$$

$$D_1 = \frac{\partial D}{\partial r} = - \mu D \tag{4.13}$$

$$D_2 = \frac{\partial D}{\partial V_E} = \frac{2D}{V_E} \tag{4.14}$$

$$N_1 = \frac{\partial N}{\partial r} = - \mu N \tag{4.15}$$

$$N_2 = \frac{\partial N}{\partial V_E} = \frac{2N}{V_E} \tag{4.16}$$

$$N_{41} = \frac{\partial N_4}{\partial r} = - \mu N_4 \tag{4.17}$$

$$N_{42} = \frac{\partial N_4}{\partial V_E} = \frac{2N_4}{V_E} \tag{4.18}$$

$$\dot{N} = N_4\dot{\alpha} + N_1\dot{r} + N_2\dot{V}_E \tag{4.19}$$

$$\dot{D} = D_1\dot{r} + D_2\dot{V}_E \tag{4.20}$$

$$\dot{N}_4 = N_{41}\dot{r} + N_{42}\,\dot{V}_E \tag{4.21}$$

$$K_{41} = \frac{\partial K_4}{\partial r} = \frac{1}{mV_E}\left[\left\{D_1 - N_{41} - \gamma(F_v - F)\right\}\cos\alpha + N_1 \sin\alpha\right] \tag{4.22}$$

$$K_{42} = \frac{\partial K_4}{\partial V_E} = \frac{1}{mV_E}\left[(D - N_{42})\cos\alpha + N_2 \sin\alpha - mK_4\right] \tag{4.23}$$

$$K_{44} = \frac{\partial K_4}{\partial \alpha} = \frac{1}{mV_E}\left[\left\{(F - D) + 2N_4\right\}\sin\alpha + N\cos\alpha\right] \tag{4.24}$$

then

$$\dot{J}_4 = \frac{1}{m}\left[- \dot{m}J_4 + \left\{\dot{N}_4 + \gamma\dot{r}\,(F_v - F) - \dot{D} - N\dot{\alpha}\right\}\sin\alpha\right.$$
$$\left. + \left\{(N_4 + F - D)\,\dot{\alpha} + \dot{N}\right\}\cos\alpha\right] \tag{4.25}$$

and

$$\dot{K}_4 = K_{44}\dot{\alpha} + K_{41}\dot{r} + K_{42}\dot{V}_E - \frac{\dot{m}}{m} K_4. \tag{4.26}$$

Substituting equations (3.29) and (3.30) for $\dot{\lambda}_3$ and $\dot{\lambda}_4$ into (4.10) and evaluating at t_o with λ_{30} and λ_{40} and using the above equations λ_{20} becomes

$$\lambda_{20} = E_1 \div E_2, \tag{4.27}$$

where

$$\begin{aligned} E_1 = - \Bigg\{ \dot{J}_4 &+ \lambda_{30} \dot{K}_4 + J_4 \Bigg[\frac{2}{mV_{E_o}} (N \sin \alpha_o + D \cos \alpha_o) \\ &+ \Bigg\{ \frac{2}{mV_{E_o}^2} \left(D \sin \alpha_o - N \cos \alpha_o + \frac{mV_{E_o}^2}{r_o} \sin \vartheta_{E_o} \right) + \frac{\dot{\vartheta}_E}{V_{E_o}} \\ &+ \frac{2\omega'}{V_{E_o}} \Bigg\} \Bigg] + K_4 \Bigg[(\omega'^2 r_o - g) \sin \vartheta_{E_o} \\ &+ \left(\frac{V_{E_o}}{r_o} - \frac{g}{V_{E_o}} + \frac{\omega'^2 r_o}{V_{E_o}} \right) \cos \vartheta_{E_o} \Bigg] \Bigg\} \end{aligned} \tag{4.28}$$

and

$$E_2 = - J_4 \cos \vartheta_{E_o} + K_4 V_{E_o} \sin \vartheta_{E_o}. \tag{4.29}$$

3. Trajectory Integration and Isolation

At a time, t_N, the equations of motion (3.20 - 3.24) and the "lambda dot" equations (3.20 - 3.32) are numerically integrated from t_N to t_{N+1}, where the variables φ_E, r, V_E, ϑ_E, λ_2, λ_3, λ_4, and λ_5

needed to start the integration are given, have been precomputed, or have been integrated from the t_{N-1} time point. The atmospheric pressure and density are determined from an atmospheric model subroutine incorporated into the program. The lift and drag coefficients, C_N' and C_{Do}, are determined by Lagrangian interpolation as functions of Mach number. The angle of attack at t_N is determined from the following subroutine.

Compute f_α: The initial α to begin the iteration is taken from the t_{N-1} time point.

If $f_\alpha \leqq$ Tol., α has been determined. (Tol $\equiv$ Tolerance)

If $f_\alpha >$ Tol., we go to equations (4.30):

$$\alpha = \alpha - \frac{f_\alpha}{f'_\alpha}, \tag{4.30}$$

where

$$f'_\alpha = \frac{\partial f_\alpha}{\partial \alpha} = -\lambda_3 \left[(F - D + 2N_4) \cos \alpha - N \sin \alpha\right]$$

$$- \frac{\lambda_4}{V_E}\left[(F - D + 2N_4) \sin \alpha + N \cos \alpha\right]. \tag{4.31}$$

If $\left|\frac{f_\alpha}{f'_\alpha}\right| \leqq$ Tol, then α has been determined.

If $\left|\frac{f_\alpha}{f'_\alpha}\right| >$ Tol, compute a new α from (4.30) using the preceding α to compute f_α and f'_α, and repeat this procedure until $\left|\frac{f_\alpha}{f'_\alpha}\right| \leqq$ Tol.

Trajectory Isolation:

A trajectory cuts off on local space-fixed circular velocity - the program may be modified to cut off on any velocity - and isolates altitude and the space-fixed flight path angle. Since the vehicle's motion is measured in an earth-fixed coordinate system, a transformation is used to compute the space-fixed values for velocity and the path angle. The relationships are

$$V^* = \left[V_E^2 + 2\omega' \; V_E r \sin \vartheta_E + \omega^2 r^2 \cos^2 \psi \right]^{1/2} \tag{4.33}$$

and

$$\vartheta^* = \tan^{-1} \left[\frac{V_E/V^* \cos \vartheta_E}{\sqrt{1 - (V_E/V^*)^2 \cos^2 \vartheta_E}} \right] ; \tag{4.34}$$

α_0 and $\dot{\alpha}_0$ are then used to isolate these end conditions. The isolation scheme given in Reference 2 is well suited for this problem.

4. Additional Equations Useful in Trajectory Analysis

The attitude of the vehicle measured from the vertical earth-fixed launch point is given by

$$\chi = \varphi_E + \vartheta_E + \alpha, \tag{4.35}$$

and the space-fixed attitude angle is given by

$$\chi^* = \chi + \omega' t. \tag{4.36}$$

The range of the vehicle measured on the surface of the earth from the earth-fixed launch point is given by

$$XXX = R_o \; \varphi_E, \tag{4.37}$$

and the range measured from the space-fixed launch point is

$$XXX^* = XXX + R_o \, \omega' t \,. \tag{4.38}$$

B. Application with $q|\alpha|$ Constraint

Input Data (In Addition to IIIA): QX, λ_{2o}, λ_{4o}. Here QX is a constant which $q|\alpha|$ cannot exceed during the trajectory; λ_{2o} and λ_{4o} are used as the isolation parameters rather than α_o and $\dot{\alpha}_o$, since the angle of attack is determined from the constraint equation and cannot be used as an isolation parameter. The computation of equations (4.9 - 4.29) is ignored in this modification of the program.

The following subroutine is used:

We compute:

$$g_1 = QX - q|\alpha|, \tag{4.39}$$

where α has been determined from the iteration of f_α.

If $g_1 \geqq 0$, $\mu_1 = 0$ and the angle of attack determined from the f_α equation are used in the main routine of the program. If $g_1 < 0$, then the angle of attack becomes

$$\alpha^* = \frac{QX \cdot \alpha}{q|\alpha|} \tag{4.40}$$

and

$$\mu_1 = \left[f_\alpha \right]_{\alpha=\alpha^*} \div m \left[q \frac{\alpha}{|\alpha|} \right]_{\alpha=\alpha^*} . \tag{4.41}$$

Equations (3.29) and (3.30) are then augmented by

$$- \mu_1 \, (\mu q |\alpha|)$$

$$\mu_1 \, (\rho V_e \, |\alpha|),$$

respectively. The constraint period ends when μ_1 goes to zero.

C. Angle of Attack Constraint

Input needed: α_c, λ_{20}, λ_{40}

The following subroutine is used:

We compute:

$$g_2 = \alpha_c^2 - \alpha^2. \tag{4.42}$$

If $g_2 \geq 0$, $\mu_1 = 0$ and the angle of attack determined from the f_α equation are used in the main routine of the program.

If $g_2 < 0$, then the angle of attack becomes

$$\alpha = \pm \alpha_c, \tag{4.43}$$

where α_c assumes the sign of α determined from the f_α equation and

$$\mu_2 = \left[f_\alpha\right]_{\alpha=\alpha_c} \div m \left[2\alpha_c\right]. \tag{4.44}$$

The constraint period ends where μ_2 goes to zero.

V. CONCLUSIONS

The optimization technique given in this report has been successfully applied to the Saturn V and Saturn IB vehicles. The equations were programmed for the IBM 7094 digital computer. The results, which are to be published later, show that an increase in orbital payload can be obtained as compared to the non-lifting first stage trajectories. This increase of orbital payload is consistent when the $q|\alpha|$ product is constrained to a reasonable value. The existing program does not contain the angle of attack constraint at the present time, but it is felt that this can be easily incorporated into the program.

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A SYSTEM OF EQUATIONS FOR OPTIMIZED POWERED FLIGHT TRAJECTORIES

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